A Two-Section Dual-Band Chebyshev Impedance Transformer

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***Abstract*—We derive a two-section dual-band impedance transformer that matches a resistive load at two arbitrary frequencies. The transformer is equivalent to a two-section Chebyshev transformer whose parameters have been adjusted to achieve reflectionless notches at the two desired frequencies.**

***Index Terms*—Chebyshev transformer, dual frequency, impedance matching, transmission line.**

I. Introduction

RECENTLY, there has been some interest in designing electrically short two-section dual-band impedance transformers that match a load at a given frequency f_1 and its first harmonic $2f_1$ [1], [2].

In this letter, we solve the more general problem of designing a two-section transformer that matches a load at two *arbitrary* frequencies, say, f_1 and f_2. We show that the transformer is equivalent to a two-section Chebyshev transformer whose parameters have been adjusted to achieve reflectionless notches at the two frequencies f_1, f_2. When $f_2 = 2f_1$, we recover the results of [1] and [2].

Possible applications are the matching of dual-band antennas operating in the cellular/PCS, GSM/DCS, WLAN, GPS, and ISM bands, and other dual-band RF applications. An M-section quarter-wavelength Chebyshev transformer has reflection response (into the main line) given in terms of the order-M Chebyshev polynomial $T_M(x)$ as follows [3]–[7]:

$$|\Gamma(f)|^2 = \frac{e_1^2 T_M^2(x)}{1 + e_1^2 T_M^2(x)}, \quad x = x_0 \cos\delta, \; \delta = \frac{\pi f}{2f_0} \tag{1}$$

where δ is the common phase length $\delta = \beta l$ of all the segments and f_0 is the frequency at which the segments are quarter-wavelength. The parameter e_1^2 is defined by

$$e_1^2 = \frac{e_0^2}{T_M^2(x_0)}, \quad e_0^2 = \frac{(Z_L - Z_0)^2}{4 Z_L Z_0}. \tag{2}$$

The reflection response $|\Gamma(f)|^2$ has a broad reflectionless band of width Δf symmetrically placed about f_0 and related to the parameter x_0 through

$$\sin\left(\frac{\pi}{4}\frac{\Delta f}{f_0}\right) = \frac{1}{x_0}. \tag{3}$$

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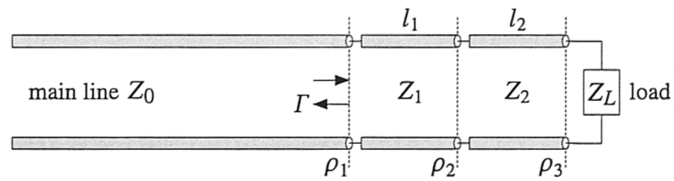

Fig. 1. Two-section impedance transformer.

The reflectionless band is mapped onto the interval $-1 \le x \le 1$ over which the Chebyshev polynomial $T_M(x)$ has M zeros and exhibits equiripple behavior. The value of the reflection response over this band is attenuated relative to its value at dc by at least the amount (in decibels)

$$A = 10\log_{10}\left(\frac{T_M^2(x_0) + e_0^2}{1 + e_0^2}\right). \tag{4}$$

The design of an M-section transformer specifies two of the three parameters $\{M, \Delta f, A\}$ and determines the third.

From the zeros of the numerator and denominator of $|\Gamma(f)|^2$ one may determine the reflection response itself, $\Gamma(f)$, and from it, through a "layer peeling" procedure, extract the reflection coefficients at the segment junctions, and then the characteristic impedances of the segments. The design details may be found in [7].

II. Design Method

A two-section transformer is depicted in Fig. 1. We will see that the desired dual-band transformer is essentially given by (1)–(4) with $M = 2$, where the second-order Chebyshev polynomial is $T_2(x) = 2x^2 - 1$. The two zeros of $T_2(x)$ are made to correspond to the two frequencies f_1, f_2.

The reflection coefficients at the three junctions are defined as usual in terms of the line, segment, and load impedances Z_0, Z_1, Z_2, Z_L (all of which are assumed to be real-valued)

$$\rho_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \quad \rho_2 = \frac{Z_2 - Z_1}{Z_2 + Z_1}, \quad \rho_3 = \frac{Z_L - Z_2}{Z_L + Z_2}. \tag{5}$$

Assuming equal travel-time segments, the reflection response may be expressed as the ratio of the z-transform polynomials [7]

$$\Gamma(f) = \frac{B(z)}{A(z)} = \frac{\rho_1 + \rho_2(1 + \rho_1\rho_3)z^{-1} + \rho_3 z^{-2}}{1 + \rho_2(\rho_1 + \rho_3)z^{-1} + \rho_1\rho_3 z^{-2}} \tag{6}$$

where z^{-1} represents the two-way travel-time delay through each section and is given by $z^{-1} = e^{-2j\delta}$, where $\delta = \pi f / 2f_0$.

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We assume that $f_1 < f_2$ and define $r = f_2/f_1$. We begin by constructing the reflection polynomial $B(z)$ to have zeros at the two desired frequencies f_1, f_2

$$B(z) = \rho_1\left(1 - e^{2j\delta_1}z^{-1}\right)\left(1 - e^{2j\delta_2}z^{-1}\right) \tag{7}$$

where

$$\delta_1 = \frac{\pi f_1}{2f_0}, \quad \delta_2 = \frac{\pi f_2}{2f_0}. \tag{8}$$

Because the coefficients of $B(z)$ in (6) are real, its zeros must be complex conjugates of each other. This can be achieved by choosing the quarter-wavelength normalization frequency f_0 to lie half-way between f_1, f_2, that is, $f_0 = (f_1 + f_2)/2 = (r + 1)f_1/2$. Then, it follows from (8) that

$$\delta_1 = \frac{\pi}{r+1}, \quad \delta_2 = r\delta_1 = \pi - \delta_1. \tag{9}$$

Thus, $e^{2j\delta_2} = e^{2j(\pi-\delta_1)} = e^{-2j\delta_1}$, and $B(z)$ takes the form

$$\begin{aligned} B(z) =& \rho_1\left(1 - e^{2j\delta_1}z^{-1}\right)\left(1 - e^{-2j\delta_1}z^{-1}\right) \\ =& \rho_1\left(1 - 2\cos 2\delta_1\, z^{-1} + z^{-2}\right). \end{aligned} \tag{10}$$

Comparing (10) with (6), we obtain the reflection coefficients

$$\rho_3 = \rho_1, \quad \rho_2 = -\frac{2\rho_1\cos 2\delta_1}{1+\rho_1^2}. \tag{11}$$

We note that the phase length δ can be expressed either in terms of f_0 or in terms of f_1

$$\delta = \frac{\pi}{2}\frac{f}{f_0} = \frac{\pi}{r+1}\frac{f}{f_1}. \tag{12}$$

Therefore, the section lengths will be quarter-wavelength at f_0 and $2(r+1)$-th wavelength at f_1

$$l_1 = l_2 = \frac{\lambda_0}{4} = \frac{\lambda_1}{2(r+1)}. \tag{13}$$

The relationship $\rho_3 = \rho_1$ is equivalent to the condition $Z_1 Z_2 = Z_L Z_0$. From (5) and (11), we obtain

$$Z_L Z_0 = Z_1 Z_2 = Z_1^2\,\frac{1+\rho_2}{1-\rho_2} = Z_1^2\,\frac{\rho_1^2 - 2\rho_1\cos 2\delta_1 + 1}{\rho_1^2 + 2\rho_1\cos 2\delta_1 + 1}.$$

Using the identity $\cos 2\delta_1 = (1-\tan^2\delta_1)/(1+\tan^2\delta_1)$ and replacing ρ_1 in terms of Z_1, we obtain the equation

$$Z_L Z_0 = Z_1^2\,\frac{Z_1^2 t_1^2 + Z_0^2}{Z_1^2 + Z_0^2 t_1^2} \tag{14}$$

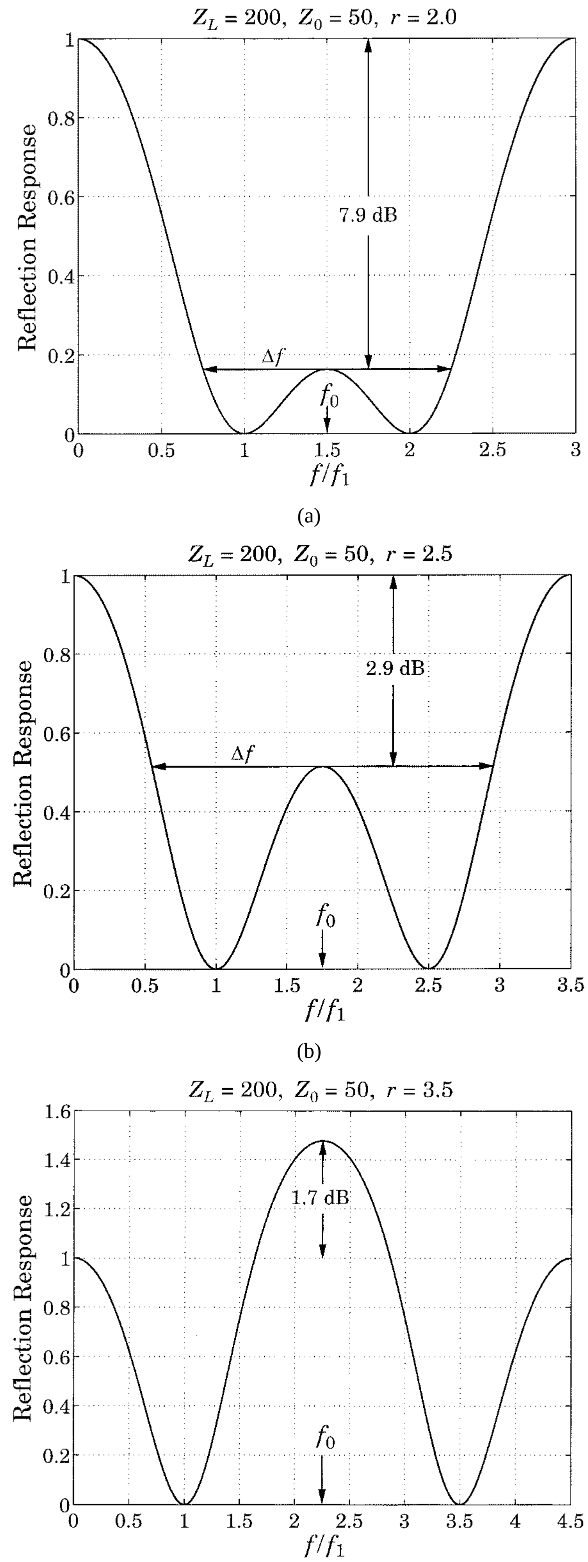
where we denoted $t_1 = \tan\delta_1$. The solution of (14) is

$$Z_1 = \sqrt{\frac{Z_0}{2t_1^2}\left[Z_L - Z_0 + \sqrt{(Z_L - Z_0)^2 + 4t_1^4 Z_L Z_0}\right]}. \tag{15}$$

Once Z_1 is known, Z_2 is obtained from $Z_2 = Z_L Z_0/Z_1$. Equations (9), (13), and (15) provide a complete solution to the two-section transformer design problem.

Next, we show that $B(z)$ is indeed proportional to the Chebyshev polynomial $T_2(x)$. Setting $z = e^{2j\delta}$, we have

$$\begin{aligned} zB(z) =& \rho_1\left(z + z^{-1} - 2\cos 2\delta_1\right) = \rho_1\left(2\cos 2\delta - 2\cos 2\delta_1\right) \\ =& 4\rho_1\left(\cos^2\delta - \cos^2\delta_1\right) = 4\rho_1\cos^2\delta_1\left(\frac{\cos^2\delta}{\cos^2\delta_1} - 1\right) \\ =& 4\rho_1\cos^2\delta_1\left(2x_0^2\cos^2\delta - 1\right) = 4\rho_1\cos^2\delta_1\,T_2(x). \end{aligned}$$



(a)

(b)

(c)

Fig. 2. Reflection responses are normalized to unity gain at dc. Notch frequencies are f_1 and $f_2 = rf_1$, with (a) $r = 2.0$, (b) 2.5, and (c) 3.5.

Therefore, $B(z) = 4\rho_1\cos^2\delta_1\,T_2(x)z^{-1}$, where we defined $x = x_0\cos\delta$ and

$$x_0 = \frac{1}{\sqrt{2}\cos\delta_1}. \tag{16}$$

We may also show that the reflection response $|\Gamma(f)|^2$ is given by (1). Using the solution (15), it can be shown after some tedious algebra that $\sigma^2 e_0^2 = 16\rho_1^2 \sin^4 \delta_1$, where $\sigma^2 = (1-\rho_1^2)(1-\rho_2^2)(1-\rho_3^2)$ and e_0^2 is given by (2). Similarly, we have $\sigma^2 e_1^2 = 16\rho_1^2 \cos^4 \delta_1$. Therefore, we may write $|B(z)|^2 = 16\rho_1^2 \cos^4 \delta_1 T_2^2(x) = \sigma^2 e_1^2 T_2^2(x)$.

On the other hand, it follows from the particular forms of the polynomials $B(z)$ and $A(z)$ in (6) that $|A(z)|^2 - |B(z)|^2 = \sigma^2$, or, $|A(z)|^2 = \sigma^2 + |B(z)|^2 = \sigma^2\left(1 + e_1^2 T_2^2(x)\right)$. Therefore

$$|\Gamma(f)|^2 = \frac{|B(z)|^2}{|A(z)|^2} = \frac{e_1^2 T_2^2(x)}{1 + e_1^2 T_2^2(x)}.$$

Thus, the reflectance is identical to that of a two-section Chebyshev transformer. However, its interpretation as a *quarter-wavelength* transformer, that is, a transformer whose attenuation at f_0 is *less* than the attenuation at dc, is valid only for a limited range of values of the parameter r, that is, $1 < r < 3$. For this range, the parameter x_0 defined in (16) is $x_0 > 1$. Then, the corresponding bandwidth about f_0 can be meaningfully defined through (3), which gives

$$\sin\left(\frac{\pi}{2(r+1)}\frac{\Delta f}{f_1}\right) = \sqrt{2}\cos\delta_1 = \sqrt{2}\cos\left(\frac{\pi}{r+1}\right). \quad (17)$$

For $1 < r < 3$, the right-hand side of (17) is always less than unity. On the other hand, when $r > 3$, the parameter x_0 becomes $x_0 < 1$, the bandwidth Δf loses its meaning, and the reflectance at f_0 becomes greater than that at dc, that is, a gain. For any value of r, the attenuation or gain at f_0 can be calculated from (4) with $M = 2$. Noting that $T_2(x_0) = 2x_0^2 - 1 = \tan^2\delta_1$, we have

$$A = 10\log_{10}\left(\frac{\tan^4\delta_1 + e_0^2}{1 + e_0^2}\right). \quad (18)$$

The quantity A is positive for $1 < r < 3$ or $\tan\delta_1 > 1$, and negative for $r > 3$ or $\tan\delta_1 < 1$.

For the special case of $r = 3$, we have $\delta_1 = \pi/4$ and $\tan\delta_1 = 1$, which gives $A = 0$. Also, it follows from (11) that $\rho_2 = 0$, which means that $Z_1 = Z_2$ and (14) gives $Z_1^2 = Z_L Z_0$. The two sections combine into a single section of double length $2l_1 = \lambda_1/4$ at f_1, that is, a single-section quarter wavelength transformer, which, as is well known, has zeros at odd multiples of its fundamental frequency.

For the case $r = 2$, we have $\delta_1 = \pi/3$ and $\tan\delta_1 = \sqrt{3}$. The design (15) reduces to that given in [2] and the section lengths become $\lambda_1/6$.

Fig. 2 shows three examples for the values $r = 2$, $r = 2.5$, and $r = 3.5$. All three transform $Z_L = 200$ into $Z_0 = 50\ \Omega$. The plotted reflectances $|\Gamma(f)|^2$ were normalized to unity gain at dc. The section impedances and attenuations A were

$$\begin{aligned} r &= 2.0, \quad Z_1 = 80.02, \quad Z_2 = 124.96, \quad A = 7.9\text{ dB} \\ r &= 2.5, \quad Z_1 = 89.02, \quad Z_2 = 112.33, \quad A = 2.9\text{ dB} \\ r &= 3.5, \quad Z_1 = 112.39, \quad Z_2 = 88.98, \quad A = -1.7\text{ dB}. \end{aligned}$$

For the cases $r = 2$ and $r = 2.5$, the bandwidth Δf calculated from (17) is depicted on the graphs. For the case $r = 3.5$, the quantity A is negative, signifying a gain at f_0. The section lengths at f_1 were in the three cases: $\lambda_1/6$, $\lambda_1/7$, and $\lambda_1/9$.

III. Conclusion

Using zero placement, we have derived a simple, electrically short, two-section dual-band impedance transformer that can match any resistive load at two arbitrary frequencies f_1, f_2 and have clarified its connection to the standard multisection quarter-wavelength Chebyshev transformer.

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